

Convergence properties of fixed-point search with general but equal phase shifts for any number of iterations

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Abstract

Grover presented the fixed-point search by replacing the selective inversions by selective phase shifts of $\pi/3$. In this paper, we investigate the convergence behavior of the fixed-point search algorithm with general but equal phase shifts for any number of iterations.

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1 Introduction

Grover's quantum search algorithm is used to find a target state in an unsorted database of size N [1][2]. The Grover's quantum search algorithm can be considered as a rotation of the state vectors in two-dimensional Hilbert space generated by the start (s) and target (t) vectors[2]. The amplitude of the target state increases monotonically towards its maximum and decreases monotonically after reaching the maximum [3]. This

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search algorithm is called the amplitude amplification algorithm. For the size $N = 2^n$ of the database, quantum search algorithm requires $O(\sqrt{N})$ steps to find the target state. As mentioned in [4] [5], unless we stop when it is right at the target state, it will drift away. A fixed-point search algorithm was presented in [4] to avoid drifting away from the target state. The fixed-point search algorithm obtained by replacing the selective inversions by selective phase shifts of $\pi/3$, converges to the target state irrespective of the number of iterations. The main advantage of the fixed-point search with equal phase shifts of $\pi/3$ is that it performs well for small but unknown initial error probability and the fixed-point behavior leads to robust quantum search algorithms [4]. However, the target state is the limit state when the number of iterations tends to the infinite.

For readability, we introduce the fixed-point search algorithm as follows. In [4] the transformation $UR_s^{\pi/3}U^\dagger R_t^{\pi/3}U$, where U is any unitary operator, was applied to the start state $|s\rangle$,

$$\begin{aligned} R_s^{\pi/3} &= I - [1 - e^{i\frac{\pi}{3}}]|s\rangle\langle s|, \\ R_t^{\pi/3} &= I - [1 - e^{i\frac{\pi}{3}}]|t\rangle\langle t|, \end{aligned} \tag{1}$$

where $|t\rangle$ stands for the target state. The transformation $UR_s^{\pi/3}U^\dagger R_t^{\pi/3}U$ is denoted as Grover's the Phase- $\pi/3$ search algorithm in [6].

Let us consider the fixed-point search algorithm with general but equal phase shifts as follows.

$$\begin{aligned} R_s^\theta &= I - [1 - e^{i\theta}]|s\rangle\langle s|, \\ R_t^\theta &= I - [1 - e^{i\theta}]|t\rangle\langle t|. \end{aligned} \tag{2}$$

The transformation $UR_s^\theta U^\dagger R_t^\theta U$ was called as the Phase- θ search algorithm and studied in [7]. It is enough to let θ be in $[0, \pi]$.

Note that if we apply U to the start state $|s\rangle$, then the amplitude of reaching the target state $|t\rangle$ is U_{ts} [2], where $||U_{ts}||^2 = 1 - \epsilon$. As indicated in [2], in the case of database search, $|U_{ts}|$ is almost $1/\sqrt{N}$, where N is the size of the database. Thus, ϵ is almost $1 - 1/N$ and ϵ is close to 1 for the large size of database.

Apply the operations U , R_s^θ , U^+ , R_t^θ , and U to the start $|s\rangle$ and let $D(\theta)$ be the deviation of the state $UR_s^\theta U^+ R_t^\theta U|s\rangle$ from the t state for any phase shifts of θ . The deviation $D(\theta)$ was reduced in [7] and is rewritten as follows.

$$D(\theta) = 4(1 - \cos \theta)^2 \epsilon(\epsilon - d)^2, \quad (3)$$

where $d = \frac{1-2\cos\theta}{2(1-\cos\theta)}$. It was shown that $D(\theta)$ is between 0 and 1 in [7]. For the Phase- $\pi/3$ search algorithm, $D(\pi/3) = \epsilon^3[4]$.

In [8], we explored the performance of the fixed-point search with general but different phase shifts for one iteration. In [7], we discussed the performance of the fixed-point search with general but equal phase shifts for one iteration.

In this paper, we investigate convergence behavior of the fixed-point search with general but equal phase shifts for any number of iterations. It is useful for designing fixed-point search algorithms for different choices of the phase shift parameter θ . The following results are established in Section 2.

- (1). The fixed-point search with equal phase shifts of $\theta \leq \pi/2$ converges to the target state.
- (2). The fixed-point search with equal phase shifts of θ , where $\pi/2 < \theta \leq \arccos(-1/4)$, converges the target state with the probability of at least 80%.
- (3). The fixed-point search with equal phase shifts of θ , where $\arccos(-1/4) < \theta \leq 2\pi/3$, converges the target state with the probability of among 66.6% and 80%.
- (4). The fixed-point search with equal phase shifts of θ , where $2\pi/3 < \theta \leq \pi$, does not converge.

In section 3, we analyze the convergence rate for different values of θ . It is demonstrated that the Phase- $\pi/3$ is not always optimal and the convergence rate can be improved by choosing $\theta > \pi/3$. In section 4, we show that for the size $N = 2^n$ of the database, $O(n)$ iterations of the Phase- θ search can find the target state. However, as indicated in [4], $O(n)$ iterations of the Phase- θ search involve the exponential queries.

2 Convergence performance of the Phase- θ search for any number of iterations

Let $\epsilon_0 = \epsilon$ and $0 < \epsilon < 1$. Then, from Eq. (3) one can obtain the following iteration equation

$$\epsilon_{m+1} = 4(1 - \cos \theta)^2 \epsilon_m (\epsilon_m - d)^2. \quad (4)$$

In this section, we discuss the convergence behavior of the Phase- θ search for any number of iterations. For the Phase- $\pi/3$ search, after recursive application of the basic iteration for m times, the failure probability $\epsilon_m = \epsilon^{3^m}$ and the success probability $|U_{m,ts}| = 1 - \epsilon^{3^m}$ [4]. The ϵ_m in Eq. (4) is the failure probability of the Phase- θ search algorithm after m iterations.

From inference [11] in [7], Eq. (4) has the following fixed-points: 0, 1 ($\theta \neq 0$), a , where $a = \cos \theta / (\cos \theta - 1)$ ($\theta \neq 0$). In other words, if the sequence $\{\epsilon_m\}$ in Eq. (4) has a limit then the limit must be 0, 1 or a . Clearly $a < d$.

To study the convergence performance for any number of iterations, we need the following results which are listed in the following paragraphs (A), (B), and (C).

(A). From Eq. (4), we obtain the following,

$$\epsilon_m - \epsilon_{m-1} = 4\epsilon_{m-1} (\cos \theta - 1)^2 (1 - \epsilon_{m-1}) (a - \epsilon_{m-1}). \quad (5)$$

Eqs. (4) and (5) imply the following convergence property.

Property 1.

(1.1) If $\epsilon_m = d$, then $\epsilon_{m+l} = 0$, for any $l > 0$.

(1.2). if $\epsilon_{m-1} > a$ and $\epsilon_{m-1} \neq 0$, $\epsilon_m < \epsilon_{m-1}$;

(1.3). If $\epsilon_{m-1} < a$ and $\epsilon_{m-1} \neq 0$, $\epsilon_m > \epsilon_{m-1}$.

(B). When $\pi/2 \leq \theta \leq \pi$, we have the following equation.

$$\epsilon_m - a = 4(\cos \theta - 1)^2 (\epsilon_{m-1} - a)(\epsilon_{m-1} - b)(\epsilon_{m-1} - c), \quad (6)$$

where $b = \frac{1}{2} - \frac{\sqrt{-\cos \theta(2-\cos \theta)}}{2(1-\cos \theta)}$, and $c = \frac{1}{2} + \frac{\sqrt{-\cos \theta(2-\cos \theta)}}{2(1-\cos \theta)}$. When $\epsilon_i = a, b$ or c , $\epsilon_{i+l} = a$ for any $l > 0$.

Note that $b < d$, $a < d$, and $d < c$.

(C). Let

$$f(x) = 4(1 - \cos \theta)^2 x(x - d)^2. \quad (7)$$

Then, the derivative of $f(x)$ is

$$f'(x) = 12(1 - \cos \theta)^2 (x - d)(x - d/3). \quad (8)$$

From Eq. (8), (1). $f'(x) = 0$ at r and d , where $r = d/3$; (2). $f'(x) < 0$ when $r < x < d$; (3). $f'(x) > 0$ when $x < r$ or $x > d$; (4). When $\theta > \pi/3$, $f(x)$ has a relative maximum $g = \frac{2(1-2\cos \theta)^3}{27(1-\cos \theta)}$ at r and a relative minimum 0 at d .

2.1 When $0 < \theta \leq \pi/2$, for any $\epsilon_0 \in (0, 1)$, the Phase- θ search converges to the target state.

Note that 0 is an attractive fixed-point when $0 < \theta < \pi/2$ and 0 is also a semi-attractive fixed-point when $\theta = \pi/2$. See inference [11] in [7].

(1). $0 < \theta \leq \pi/3$

For this case, $d \leq 0$ and $a < 0$. In Eq. (4), $d = 0$ means $\epsilon_{m+1} = \epsilon_m^3$, which is Grover's Phase- $\pi/3$ search. From $d < 0$ and Eq. (4), $\epsilon_m > 0$. By property (1.2), always $\epsilon_m < \epsilon_{m-1}$ when $0 < \theta \leq \pi/3$. That is, the sequence $\{\epsilon_m\}$ in Eq. (4) decreases monotonically. Therefore, for any ϵ_0 in $(0, 1)$ $\lim_{m \rightarrow \infty} \epsilon_m = 0$.

(2). $\pi/3 < \theta \leq \pi/2$

For this case, $a \leq 0$, $0 < d \leq 1/2$. Hence, from Eq. (4) $0 \leq \epsilon_i < 1$. By property (1.2), always $\epsilon_m \leq \epsilon_{m-1}$ when $\pi/3 < \theta \leq \pi/2$. That is, the sequence $\{\epsilon_m\}$ in Eq. (4) decreases. Factually, the sequence $\{\epsilon_m\}$ in Eq. (4) decreases monotonically and $\epsilon_m > 0$, or is of the form $\epsilon_0 > \epsilon_1 > \dots > \epsilon_k = 0$ and $\epsilon_l = 0$ for any $l > k$. Therefore, for any ϵ_0 in $(0, 1)$ $\lim_{m \rightarrow \infty} \epsilon_m = 0$.

Example 1. For the Phase- $\pi/2$ search, $\epsilon_m = \epsilon_{m-1}(2\epsilon_{m-1} - 1)^2$. Let $\epsilon_0 = 0.99999$. See Fig. 1.

$$\begin{aligned}\epsilon_1 &= 0.999\,95, \quad \epsilon_2 = 0.999\,75, \quad \epsilon_3 = 0.998\,75, \quad \epsilon_4 = 0.993\,76, \\ \epsilon_5 &= 0.969\,11, \quad \epsilon_6 = 0.853\,07, \quad \epsilon_7 = 0.425\,37, \quad \epsilon_8 = 9.476\,6 \times 10^{-3}.\end{aligned}$$

2.2 When $\pi/2 < \theta < \arccos(-1/4)$, the Phase- θ search converges the target state with the success probability of $(1 - a) > 80\%$.

For the Phase- θ search, $a < g < r < b < d < c$. Note that a is an attractive fixed-point. See inference [11] in [7]. From Eq. (6), we have the following property.

Property 2.

$$(2.1). \quad a < \epsilon_m \leq g \text{ whenever } a < \epsilon_{m-1} < b;$$

$$(2.2). \quad 0 \leq \epsilon_m < a \text{ whenever } b < \epsilon_{m-1} < c \text{ or } \epsilon_{m-1} < a.$$

The convergence region of the Phase- θ search

(A). When $\epsilon_0 \in (0, c]$ and $\epsilon_0 \neq d$, the deviation from the target state converges to the fixed-point a .

There are four cases. The argument is the following.

Case 1. When $\epsilon_0 = a$ or b or c , it is trivial by Eq. (6).

Case 2. $\epsilon_0 < a$. By property (2.2), $0 < \epsilon_m < a$ for any m . By property 1, the sequence $\{\epsilon_m\}$ increases monotonically. Hence, the sequence $\{\epsilon_m\}$ converges to a from below.

Case 3. $a < \epsilon_0 < b$. By property (2.1), always $a < \epsilon_m \leq g$ for any $m > 0$, and by property 1, the sequence $\{\epsilon_m\}$ decreases monotonically. Hence, the sequence $\{\epsilon_m\}$ converges to a from above.

Case 4. $b < \epsilon_0 < c$ and $\epsilon_0 \neq d$. By property (2.2), $0 < \epsilon_1 < a$. Then, it turns to case 2.

Conclusively, when $\epsilon_0 \in (0, c]$ and $\epsilon_0 \neq d$, from the above four cases, $\epsilon_m \neq d$, hence $\lim_{m \rightarrow \infty} \epsilon_m = a$.

(B). When $\epsilon_0 \in (c, 1)$, the deviation from the target state converges to the fixed-points a or 0 .

By property (1.2), $\epsilon_0 > \dots > \epsilon_{j^*-1} (> c) > \epsilon_{j^*} (\leq c)$. If $\epsilon_{j^*} = d$, then $\epsilon_m = 0$ for any $m > j^*$. Otherwise, $\lim_{m \rightarrow \infty} \epsilon_m = a$ by the above (A).

2.3 Phase-arccos($-1/4$) search converges the target state with the success probability of 80%.

For the Phase-arccos($-1/4$) search, $a = 1/5$ is an attractive fixed-point, see inference [11] in [7]. $b = a = 1/5$, $d = 3/5$, and $c = 4/5$. The iteration equation is $\epsilon_m = \epsilon_{m-1}(5\epsilon_{m-1} - 3)^2/4$. Eq. (6) becomes the following.

$$\epsilon_m - 1/5 = \frac{25}{4} (\epsilon_{m-1} - 4/5) (\epsilon_{m-1} - 1/5)^2. \quad (9)$$

From Eq. (9) we have the following property.

Property 3.

(3.1). $\epsilon_m < 1/5$ when $\epsilon_{m-1} < 4/5$ and $\epsilon_{m-1} \neq 1/5$.

(3.2). $\epsilon_m > 1/5$ when $\epsilon_{m-1} > 4/5$.

The convergence region of the Phase-arccos($-1/4$) search

(A). When $\epsilon_0 \in (0, 4/5]$ and $\epsilon_0 \neq 3/5$, the deviation from the target state converges to the fixed-point $1/5$.

When $\epsilon_0 = 1/5$ or $4/5$, it is trivial by Eq. (6). When $\epsilon_0 \in (0, 4/5)$ and $\epsilon_0 \neq 1/5$, always $\epsilon_m < 1/5$ for $m > 0$ by property (3.1) and the sequence $\{\epsilon_m\}$ increases monotonically from $m > 0$ by property (1.3). Therefore, the sequence $\{\epsilon_m\}$ converges to $1/5$ from below.

(B) When $\epsilon_0 \in (4/5, 1)$, the deviation from the target state converges to the fixed-points $1/5$ or 0 .

By property (1.2), $\epsilon_0 > \epsilon_1 > \dots > \epsilon_m (\leq 4/5)$. Case 1. If $\epsilon_m = 3/5$, then $\epsilon_i = 0$ for any $i > m$. Case 2. Otherwise, by the above (A), $\lim_{m \rightarrow \infty} \epsilon_m = 1/5$.

Example 2. Let $\epsilon_0 = 0.9999$;

$$\epsilon_1 = 0.9994, \quad \epsilon_2 = 0.9964, \quad \epsilon_3 = 0.97855, \quad \epsilon_4 = 0.87641,$$

$$\epsilon_5 = 0.41850, \quad \epsilon_6 = 8.6165 \times 10^{-2}, \quad \epsilon_7 = 0.14219, \quad \epsilon_8 = 0.18626,$$

$$\epsilon_9 = 0.19928, \quad \epsilon_{10} = 0.2.$$

2.4 When $\arccos(-1/4) < \theta \leq 2\pi/3$, the Phase- θ search converges the target state with the success probability of $(1 - a)$, where $66\% \leq (1 - a) < 80\%$.

For the Phase- θ search, $b < r < a < g < d < c$. Note that a is an attractive fixed-point when $\arccos(-1/4) < \theta < 2\pi/3$ and $1/3$ is a semi-attractive fixed-point when $\theta = 2\pi/3$. See inference [11] in [7]. From Eq. (6), we have the following property.

Property 4.

$$(4.1). \ a < \epsilon_m \leq g \text{ whenever } b < \epsilon_{m-1} < a;$$

$$(4.2). \ 0 \leq \epsilon_m < a \text{ whenever } a < \epsilon_{m-1} < c \text{ or } \epsilon_{m-1} < b.$$

The convergence region of the Phase- θ search

(A). When $\epsilon_0 \in (0, c]$ and $\epsilon_0 \neq d$, the deviation from the target state converges to the fixed-point a .

There are seven cases. We argue them as follows.

Case 1. If $\epsilon_0 = a$ or b or c , then it is trivial.

Case 2. $a < \epsilon_0 \leq g$. The proof is put in Appendix A.

Case 3. $\epsilon_0 < b$. By property (1.3), ϵ_j increases monotonically from ϵ_0 until $\epsilon_{j^*-1} < b$ and $b \leq \epsilon_{j^*} < f(b) = a$ since $f'(x) > 0$ when $x < b$. If $\epsilon_{j^*} = b$, it is trivial. Otherwise, by property (4.1), ϵ_{j^*+1} is in $(a, g]$.

Now it turns to case 2.

Case 4. $b < \epsilon_0 \leq r$. When $\epsilon_0 = r$, $\epsilon_m = f^{(m-1)}(g)$. From the proof of case 2, $\lim_{m \rightarrow \infty} f^{(m-1)}(g) = a$. Next consider that $b < \epsilon_0 < r$. Since $f'(x) > 0$ when $b < x < r$, $f(b) < f(\epsilon_0) < f(r)$. That is, $a < \epsilon_1 < g$. It turns to case 2.

Case 5. $r < \epsilon_0 < a$. Since $f'(x) < 0$ when $r < x < a$, $a < \epsilon_1 < g$. It turns to case 2.

Case 6. $g < \epsilon_0 < d$. Since $f'(x) < 0$ when $g < x < d$ and $a < g$, $0 < \epsilon_1 < f(g) < a$. Then, it turns to cases 1, 3, 4, 5.

Case 7. $d < \epsilon_0 < c$. Since $f'(x) > 0$ when $d < x < c$, $0 < \epsilon_1 < a$. Then, it turns to cases 1, 3, 4, 5.

(B). When $\epsilon_0 \in (c, 1)$, the deviation from the target state converges to the fixed-points a or 0 .

When $\epsilon_0 > c$, by property (1.2) the sequence $\{\epsilon_i\}$ decreases monotonically from ϵ_0 to $\epsilon_{i^*} \leq c$. Case 1, if $\epsilon_{i^*} = d$, then $\epsilon_i = 0$ for any $i > i^*$. Case 2. Otherwise, by the above (A) $\lim_{m \rightarrow \infty} \epsilon_m = a$.

Example 3. For the Phase- $2\pi/3$ search, $a = 1/3$. The iteration equation becomes $\epsilon_m = \epsilon_{m-1}(3\epsilon_{m-1} - 2)^2$.

Let $\epsilon_0 = 0.99999$. We have the following iterations. See Fig. 1.

$$\begin{aligned}\epsilon_1 &= 0.999\,93, \quad \epsilon_2 = 0.999\,51, \quad \epsilon_3 = 0.996\,57, \quad \epsilon_4 = 0.976\,17, \\ \epsilon_5 &= 0.841\,59, \quad \epsilon_6 = 0.231\,76, \quad \epsilon_7 = 0.394\,52, \quad \epsilon_8 = 0.263\,50, \\ \epsilon_9 &= 0.385\,47, \quad \epsilon_{10} = 0.274\,32, \quad \epsilon_{11} = 0.380\,05, \quad \epsilon_{12} = 0.280\,99, \\ \epsilon_{13} &= 0.376\,17.\end{aligned}$$

2.5 $2\pi/3 < \theta \leq \pi$, the Phase- θ search does not converge.

For the Phase- θ search, $b < r < a < d < c$. From Eq. (6), we have the following property.

Property 5

$$(5.1). \quad a < \epsilon_m \leq g \text{ whenever } b < \epsilon_{m-1} < a;$$

$$(5.2). \quad 0 \leq \epsilon_m < a \text{ whenever } a < \epsilon_{m-1} < c \text{ or } \epsilon_{m-1} < b.$$

For large ϵ , by property (1.2), the sequence $\{\epsilon_i\}$ decreases monotonically from ϵ_0 to $\epsilon_{i^*} (\leq c)$. If $\epsilon_{i^*} = d$, then $\epsilon_i = 0$ for any $i > i^*$. If $\epsilon_{i^*} = a, b$, or c , then $\epsilon_i = a$ when $i > i^*$. Otherwise, when $i > i^*$, ϵ_i oscillate around the fixed point a by property 1. However, the sequence $\{\epsilon_i\}$ does not converges because $a, 0$ and 1 are repulsive fixed-points.

Example 4. For the Phase- π search, the iteration equation becomes $\epsilon_m = \epsilon_{m-1}(4\epsilon_{m-1} - 3)^2$, $a = 1/2$.

Let $\epsilon = 0.99999$. We have the following iterations. See Fig. 1.

$$\begin{aligned}\epsilon_1 &= 0.999\,91, \quad \epsilon_2 = 0.999\,19, \quad \epsilon_3 = 0.992\,73, \quad \epsilon_4 = 0.935\,83, \\ \epsilon_5 &= 0.517\,07, \quad \epsilon_6 = 0.448\,87, \quad \epsilon_7 = 0.651\,25, \quad \epsilon_8 = 0.101\,61, \\ \epsilon_9 &= 0.683\,49, \quad \epsilon_{10} = 4.837\,6 \times 10^{-2}.\end{aligned}$$

Clearly, the sequence $\{\epsilon_i\}$ monotonically decreases from ϵ_0 to ϵ_6 . Note that after the sixth iteration, ϵ_m oscillate around the fixed point $1/2$.

3 A comparison of rates of convergence after any number of iterations

For the Phase- $\pi/3$ search, let the iteration equation be $\epsilon_m(\pi/3) = (\epsilon_{m-1}(\pi/3))^3$, where $\epsilon_m(\pi/3)$ is the failure probability of the Phase- $\pi/3$ search algorithm after m iterations. For the Phase- θ search, we can rewrite Eq. (4) as $\epsilon_m(\theta) = 4(1 - \cos \theta)^2 \epsilon_{m-1}(\theta)(\epsilon_{m-1}(\theta) - d)^2$, where the $\epsilon_m(\theta)$ is the failure probability of the Phase- θ search algorithm after m iterations. We want to compare the failure probability of the Phase- θ ($\neq \pi/3$) search algorithm with the one of the Phase- $\pi/3$ search after m iterations. It is known that the less the failure probability is, the faster the algorithm converges. By factoring,

$$\begin{aligned} \epsilon_m(\theta) - \epsilon_m(\pi/3) &= \\ &= \epsilon_{m-1}(\theta)(2 \cos \theta - 1)(1 - \epsilon_{m-1}(\theta))(3 - 2 \cos \theta) * \\ &= (\epsilon_{m-1}(\theta) - \frac{1 - 2 \cos \theta}{3 - 2 \cos \theta}) + \epsilon_{m-1}^3(\theta) - \epsilon_{m-1}^3(\pi/3). \end{aligned} \quad (10)$$

We have the following results.

(1). $\pi/3 < \theta \leq \pi$

Case 1. For large ϵ , the Phase- θ search converges faster than the Phase- $\pi/3$ search for m iterations until $\epsilon_{m-1}(\theta) < \frac{1-2 \cos \theta}{3-2 \cos \theta}$.

In [7], we show if $\epsilon_0(\theta) = \epsilon_0(\pi/3) = \epsilon > \frac{1-2 \cos \theta}{3-2 \cos \theta}$ then $\epsilon_1(\theta) < \epsilon_1(\pi/3) = \epsilon^3$. If $\epsilon_{m-1}(\theta) > \frac{1-2 \cos \theta}{3-2 \cos \theta}$ and $\epsilon_{m-1}(\theta) < \epsilon_{m-1}(\pi/3)$, then by Eq. (10) $\epsilon_m(\theta) < \epsilon_m(\pi/3)$. Thus, $\epsilon_i(\theta) < \epsilon_i(\pi/3)$, where $i = 1, 2, \dots, m-1$, until $\epsilon_{m-1}(\theta) < \frac{1-2 \cos \theta}{3-2 \cos \theta}$. It says that after m iterations, the failure probability of the Phase- θ search is less than the one of the Phase- $\pi/3$ search until $\epsilon_{m-1}(\theta) < \frac{1-2 \cos \theta}{3-2 \cos \theta}$. It suggests us first to use the fixed-point search with large phase shifts for the large size of database.

Case 2. For small ϵ , the Phase- $\pi/3$ search converges faster than the Phase- θ search for m iterations until $\epsilon_{m-1}(\theta) > \frac{1-2 \cos \theta}{3-2 \cos \theta}$.

In [7], we show if $\epsilon_0(\theta) = \epsilon_0(\pi/3) = \epsilon < \frac{1-2 \cos \theta}{3-2 \cos \theta}$ then $\epsilon_1(\theta) > \epsilon_1(\pi/3) = \epsilon^3$. If $\epsilon_{m-1}(\theta) < \frac{1-2 \cos \theta}{3-2 \cos \theta}$ and $\epsilon_{m-1}(\theta) > \epsilon_{m-1}(\pi/3)$, then by Eq. (10) $\epsilon_m(\theta) > \epsilon_m(\pi/3)$.

(2). When $0 < \theta < \pi/3$, the Phase- $\pi/3$ search converges faster than the Phase- θ search for any ϵ for any number of iterations.

When $\epsilon_0(\theta) = \epsilon_0(\pi/3) = \epsilon$, in [7] we show $\epsilon_1(\theta) > \epsilon_1(\pi/3)$. Assume that $\epsilon_{m-1}(\theta) > \epsilon_{m-1}(\pi/3)$. From Eq. (10), it is easy to see that $\epsilon_m(\theta) > \epsilon_m(\pi/3)$. Therefore, $\epsilon_m(\theta) > \epsilon_m(\pi/3)$ for any m . Hence, when $0 < \theta < \pi/3$, the Phase- $\pi/3$ search converges faster than the Phase- θ search for any ϵ for any number of iterations.

4 For any known ϵ , $O(n)$ iterations can find the target state.

Assume that a database has $N = 2^n$ states (items). Then a state (an item) is found with the probability of $1/N$ [2]. In other words, the failure probability $\epsilon = 1 - 1/N$. It is known that the Phase- $\pi/3$ search converges the target state. In this section, we investigate how to use the fixed-point search to find the target state in a database when ϵ is known. As discussed in [4], the fixed-point search is a recursive algorithm, therefore the number of queries grows exponentially with the number of recursion levels. For example, the Phase- $\pi/3$ search at i -level recursion involves $q_i = (3^i - 1)/2$ queries [6]. This implies that $O(n)$ iterations of the Phase- θ search involve the exponential queries.

4.1 When $\epsilon \leq 3/4$, only one iteration is needed to find the target state.

When $0 \leq \epsilon \leq \frac{3}{4}$, $\left|1 - \frac{1}{2(1-\epsilon)}\right| \leq 1$. Let $\cos \theta = 1 - \frac{1}{2(1-\epsilon)}$. Then $D(\theta) = 0$. Therefore, if ϵ is fixed and $0 \leq \epsilon \leq \frac{3}{4}$, then we choose $\theta = \arccos[1 - \frac{1}{2(1-\epsilon)}]$, which is in $(\pi/3, \pi]$, as phase shifts. The Phase- $\arccos[1 - \frac{1}{2(1-\epsilon)}]$ search will obviously make the deviation vanish. It means that one iteration will reach t state if the θ is chosen as phase shifts. Ref. [7].

4.2 When $\epsilon > 3/4$, $O(n)$ iterations can find the target state.

4.2.1 First use the Phase- $\pi/3$ search

For the Phase- $\pi/3$ search, $\epsilon_n = \epsilon^{3^n}$. There exists the least natural number n^* such that $\epsilon^{3^{n^*}} \leq 3/4$. By calculating, $n^* = \lceil (\ln \ln \frac{4}{3} - \ln \ln \frac{1}{\epsilon}) / \ln 3 \rceil$.

Lemma 1. For the Phase- $\pi/3$ search, $n^* = O(n)$.

Proof. In the case of database search, Let $N = 2^n$. Then $\epsilon = 1 - 2^{-n}$, and $\lim_{n \rightarrow +\infty} \frac{n^*}{n} = \frac{\ln 2}{\ln 3}$. Almost $\frac{n \ln 2}{\ln 3} \approx [0.63n]$. Let $N = 10^n$. Then $\epsilon = 1 - 10^{-n}$, and $\lim_{n \rightarrow +\infty} \frac{n^*}{n} = \frac{\ln 10}{\ln 3} = \frac{1}{\lg 3}$. Almost $\frac{n}{\lg 3} \approx 2n$. Thus, $n^* = O(n)$. Hence, when $\epsilon > 3/4$, after n^* iterations of the Phase- $\pi/3$ search the failure probability $\epsilon_{n^*} \leq 3/4$. Then, after one iteration of the Phase- $\arccos[1 - \frac{1}{2(1-\epsilon_{n^*})}]$ search by using the result in section 4.1, it will reach t state.

Example 5. Let $N = 10^4$. Then $\epsilon = 1 - 10^{-4}$, $n^* = 8$, $\epsilon_7 = 0.80332$, $\epsilon_8 = 0.5184$. See Fig.1. However, for this purpose, it only needs 4 iterations for the Phase- π search. See example 7.

Example 6. Let $N = 2^{10}$. Then $\epsilon = 1 - 2^{-10}$, $n^* = 6$, $\epsilon_5 = 0.78856$, $\epsilon_6 = 0.49035$.

4.2.2 First use the Phase- θ ($\neq \pi/3$) search

Let $\epsilon > 3/4$. Then, by property (1.2), for the Phase- θ search, there exists the least natural number $m^*(\theta)$ such that $\epsilon_0 > \epsilon_1 > \dots > \epsilon_{m^*(\theta)-1} (> 3/4) > \epsilon_{m^*(\theta)} (\leq 3/4) \dots$. Thus, after $m^*(\theta)$ iterations of the Phase- θ search, the failure probability $\epsilon_{m^*(\theta)} \leq 3/4$. Then, after one iteration for the Phase- $\arccos[1 - \frac{1}{2(1-\epsilon_{m^*(\theta)})}]$ search by using the result in section 4.1, it will reach t state.

Next let us calculate $m^*(\theta)$. Let $\delta = 1 - \epsilon$, where δ is the success probability. When ϵ is close to 1, δ is close to 0. Then, for large ϵ , by induction $\epsilon_l = 1 - [1 + 4(1 - \cos \theta)]^l \delta + O(\delta^2)$. Thus, $\epsilon_l \approx 1 - [1 + 4(1 - \cos \theta)]^l \delta$. By this approximate formula of ϵ_l , $m^*(\theta) \approx M^*(\theta) = \lceil \frac{-2 \lg 2 - \lg \delta}{\lg(1 + 4(1 - \cos \theta))} \rceil$.

In the case of database search, let $N = 2^n$. Then $\epsilon = 1 - 2^{-n}$, $\delta = 2^{-n}$, and $m^*(\theta) \approx M^*(\theta) = \lceil \frac{(n-2) \lg 2}{\lg(1 + 4(1 - \cos \theta))} \rceil$. For the Phase- $\pi/3$ search, $M^*(\pi/3) = \lceil \frac{n \ln 2}{\ln 3} - \frac{2 \ln 2}{\ln 3} \rceil$. Note that $\frac{2 \ln 2}{\ln 3} = 1.2619$. Therefore, when n is large enough $M^*(\pi/3) \approx m^*(\pi/3) = n^*$. For the Phase- π search, $m^*(\pi) \approx M^*(\pi) = \lceil (n - 2) \lg 2 / (2 \lg 3) \rceil \approx (\lg 2)n$. See Table (I).

Let $N = 10^n$. Then $\epsilon = 1 - 10^{-n}$, $\delta = 10^{-n}$, and $m^*(\theta) \approx M^*(\theta) = \lceil \frac{n-2\lg 2}{\lg(1+4(1-\cos\theta))} \rceil$. For the Phase- $\pi/3$ search, $M^*(\pi/3) = \lceil \frac{n}{\lg 3} - \frac{2\lg 2}{\lg 3} \rceil$. Note that $\frac{2\lg 2}{\lg 3} = \frac{2\ln 2}{\ln 3}$. Therefore, when n is large enough $M^*(\pi/3) \approx m^*(\pi/3) = n^*$. For the Phase- π search, $m^*(\pi) \approx M^*(\pi) = \lceil (n - 2\lg 2)/(2\lg 3) \rceil \approx n$. See

Example 7. Let $N = 10^4$. Then $\epsilon = 1 - 10^{-4}$, $M^*(\pi) = 4$, $\epsilon_4 = 0.47532$. See Table (II).

Lemma 2. For the Phase- θ ($\neq \pi/3$) search, $m^*(\theta) = O(n)$.

Proof. When $\pi/3 < \theta \leq \pi$, as discussed in case 1 of (1) in Sec. 3, $m^*(\theta) < n^*$. By lemma 1, this lemma holds. When $0 < \theta < \pi/3$, from the approximate formula of $m^*(\theta)$, $m^*(\theta) = O(n)$.

Remark. $M^*(\theta)$ monotonically decreases as θ increases from 0 to π , especially $\frac{M^*(\pi)}{n^*} \approx 1/2$. Therefore, we suggest first to use Phase- π search for $m^*(\pi)$ times to get the failure probability $\epsilon_{m^*} \leq 3/4$.

5 Summary

In this paper, we investigate convergence performance of the Phase- θ search for any number of iterations. We discuss the convergence region and rate of the Phase- θ search and study the convergence behavior of the Phase- θ search for different initial ϵ_0 .

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We want to thank the reviewer of [7] for suggesting us to study the convergence behavior of the fixed-point search with general but equal phase shifts for any number of iterations.

6 Appendix A

Proof. Since $f'(x) < 0$ when $r < x < d$, $f(g) \leq \epsilon_1 < a$. Note that $r < f(g)$. Thus, $r < f(g) \leq \epsilon_1 < a$. Let $f^{(k)}(x) = f(f^{(k-1)}(x))$. Since $f'(x) < 0$, $a < \epsilon_2 \leq f^{(2)}(g) < f(r) = g$ and $r < f(g) < f^{(3)}(g) \leq \epsilon_3 < a$. By induction, generally $a < \epsilon_{2k} \leq f^{(2k)}(g) < f^{(2k-2)}(g) < \dots f^{(2)}(g) < g$ and $r < f(g) < \dots < f^{(2k-1)}(g) < f^{(2k+1)}(g) \leq \epsilon_{2k+1} < a$. That is, ϵ_i oscillate around the fixed point a by property 1 and between $f^{(2k)}(g)$ and $f^{(2k+1)}(g)$. It is plain that the sequence $\{f^{(2k)}(g)\}$ decreases monotonically as k increases while the sequence $\{f^{(2k+1)}(g)\}$ increases monotonically as k does. Hence, the sequences $\{f^{(2k)}(g)\}$ and $\{f^{(2k+1)}(g)\}$

have limits. Let $\lim_{k \rightarrow \infty} f^{(2k)}(g) = \alpha$ and $\lim_{k \rightarrow \infty} f^{(2k+1)}(g) = \beta$. Clearly, $\alpha, \beta < d$. From Eq. (4), $f^{(2k)}(g) = 4(1 - \cos \theta)^2 f^{(2k-1)}(g)(f^{(2k-1)}(g) - d)^2$ and $f^{(2k+1)}(g) = 4(1 - \cos \theta)^2 f^{(2k)}(g)(f^{(2k)}(g) - d)^2$. By taking the limits, we obtain $\alpha = 4(1 - \cos \theta)^2 \beta(\beta - d)^2$ and $\beta = 4(1 - \cos \theta)^2 \alpha(\alpha - d)^2$. By substituting, $\beta = [4(1 - \cos \theta)^2]^2 \beta(\beta - d)^2(\alpha - d)^2$. By cancelling, $[4(1 - \cos \theta)^2]^2(\beta - d)^2(\alpha - d)^2 = 1$. Then, there are two cases. Case 1. $4(1 - \cos \theta)^2(d - \beta)(d - \alpha) = 1$. By solving this equation, $\alpha = \beta = 1$ or $\alpha = \beta = a$. Since $\alpha, \beta < d < 1$, then $\alpha = \beta = a$. Case 2. $4(1 - \cos \theta)^2(d - \beta)(d - \alpha) = -1$. There is no solution because $\alpha, \beta < d$. Therefore, $\lim_{k \rightarrow \infty} f^{(2k)}(g) = \lim_{k \rightarrow \infty} f^{(2k+1)}(g) = a$. Then, $\lim_{k \rightarrow \infty} f^{(k)}(g) = a$, and also $\lim_{m \rightarrow \infty} \epsilon_m = a$. We finish the proof.

References

- [1] L.K.Grover, Phys. Rev. Lett. 79 (1997) 325.
- [2] L.K.Grover, Phys. Rev. Lett. 80 (1998) 4329.
- [3] D. Li et al., Theor. Math. Phys. 144(3) (2005) 1279-1287.
- [4] L.K.Grover, Phys. Rev. Lett. 95 (2005) 150501.
- [5] G. Brassard, Science 275 (1997) 627.
- [6] T.Tulsi, L. Grover, and A. Patel, quant-ph/0505007. Also, Quant. Inform. and Comput. 6(6) (2006) 483-494.
- [7] D. Li et al., Eur. Phys. J. D 45 (2007) 335-340.
- [8] D. Li et al., Phys. Lett. A 362 (2007) 260-264. Also see quant-ph/0604062.